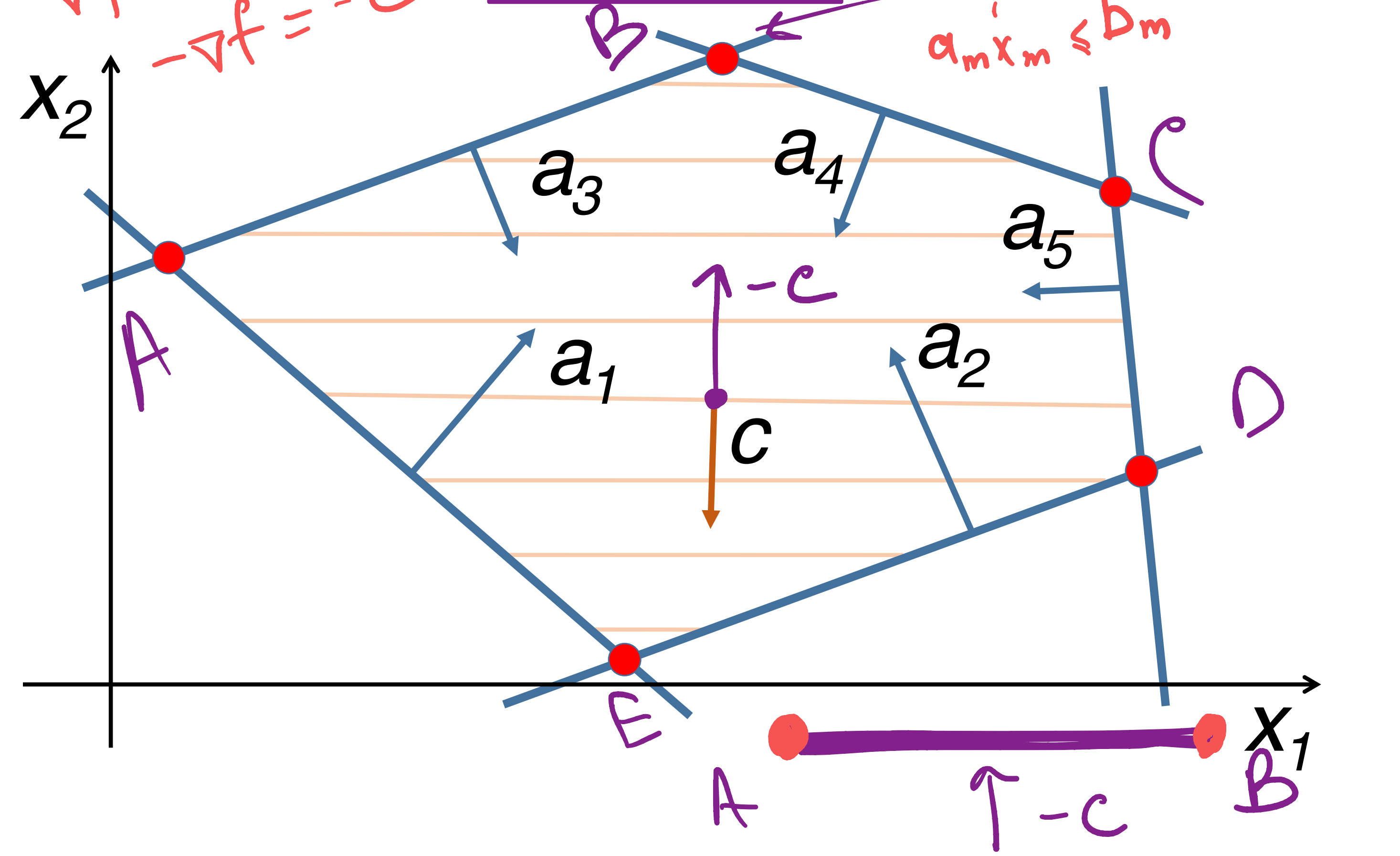
Introduction to Linear Programming

What is LP

Generally speaking, all problems with linear objective and linear equalities\inequalities constraints could be considered as Linear Programming. However, there are some widely accepted formulations.

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} c^\top x \\
\text{s.t. } &\underset{m\times n}{A}x \leq \underset{m\times 1}{b}
\end{aligned}
\tag{LP.Basic}
$$



for some vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and matrix $A \in \mathbb{R}^{m \times n}$. Where the inequalities are interpreted component-wise.

Standard form

This form seems to be the most intuitive and geometric in terms of visualization. Let us have vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and matrix $A \in \mathbb{R}^{m \times n}$.

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} c^\top x \\
\text{s.t. } &Ax = b \\
&x_i \geq 0, \; i = 1, \dots, n
\end{aligned}
\tag{LP.Standard}
$$

Canonical form

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} c^\top x \\
\text{s.t. } & Ax \leq b \\
& x_i \geq 0, \; i = 1, \dots, n
\end{aligned}
\tag{LP.Canonical}
$$

Real world problems

Diet problem

Imagine, that you have to construct a diet plan from some set of products: 🍌🍰🍗🥚🐟. Each of the products has its own vector of nutrients. Thus, all the food information could be processed through the matrix W. Let also assume, that we have the vector of requirements for each of nutrients $r \in \mathbb{R}^n$. We need to find the cheapest configuration of the diet, which meets all the requirements:

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^p} c^\top x \\
\text{s.t. } & Wx \geq r \\
& x_i \geq 0, \; i = 1, \dots, n
\end{aligned}
$$

p products

Requirements

$\boxed{W \in \mathbb{R}^{n \times p}},$

$r \in \mathbb{R}^n$

$$
\begin{array}{r}
\text{Proteins} \\
\text{Carbs} \\
\text{Fats} \\
\text{Calories} \\
\text{Vitamin D}
\end{array}
\begin{bmatrix}
8 \\
12 \\
3 \\
100 \\
3
\end{bmatrix}
\qquad
\begin{bmatrix}
\\
\\
\\
\\
\
\end{bmatrix}
$$

$\boxed{c \in \mathbb{R}^p}$ - cost per 100 g

n nutrients

$$
\min_{x \in \mathbb{R}^p} c^\top x
$$
$$
\boxed{Wx \geq r}
$$

How to retrieve LP

Basic transformations

Inequality to equality by increasing the dimension of the problem by m.

$$
Ax \leq b \leftrightarrow \begin{cases} Ax + z = b \\ z \geq 0 \end{cases}
$$

unsigned variables to nonnegative variables.

$$
x \leftrightarrow \begin{cases} x = x_+ - x_- \\ x_+ \geq 0 \\ x_- \geq 0 \end{cases}
$$

Chebyshev approximation problem

$$
\min_{x \in \mathbb{R}^n} \|Ax - b\|_\infty \leftrightarrow \min_{x \in \mathbb{R}^n} \max_{i} |a_i^\top x - b_i|
$$

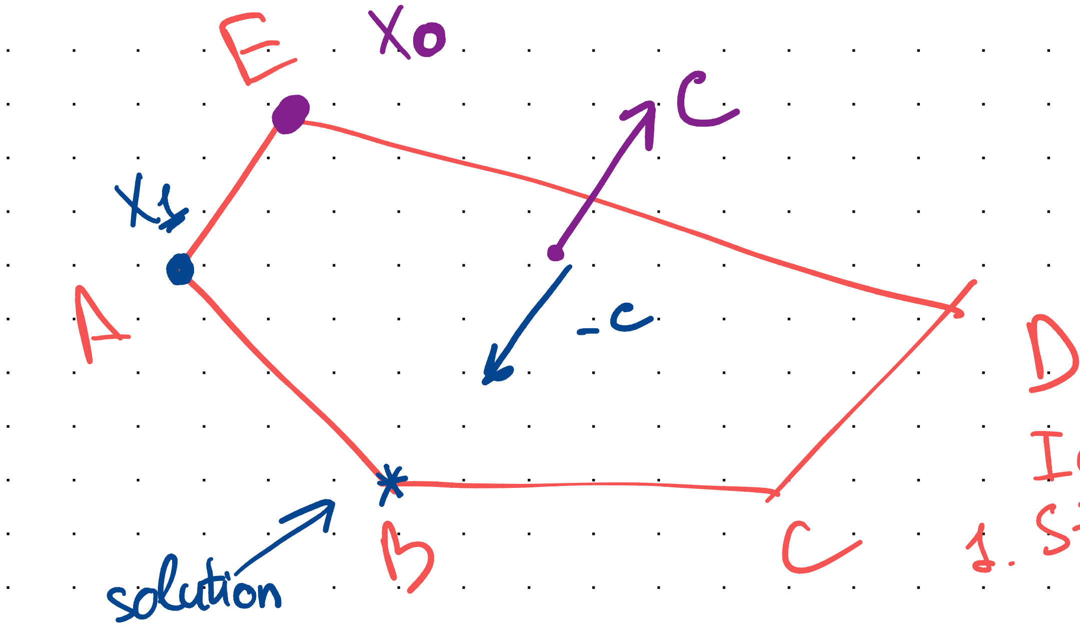
$$
\begin{aligned}
&\min_{t \in \mathbb{R}, x \in \mathbb{R}^n} t \\
\text{s.t. } & a_i^\top x - b_i \leq t, \; i = 1, \dots, n \\
& -a_i^\top x + b_i \leq t, \; i = 1, \dots, n
\end{aligned}
$$

l_1 approximation problem

$$
\min_{x \in \mathbb{R}^n} \|Ax - b\|_1 \leftrightarrow \min_{x \in \mathbb{R}^n} \sum_{i=1}^n |a_i^\top x - b_i|
$$

$$
\begin{aligned}
&\min_{t \in \mathbb{R}^n, x \in \mathbb{R}^n} \mathbf{1}^\top t \\
\text{s.t. } & a_i^\top x - b_i \leq t_i, \; i = 1, \dots, n \\
& -a_i^\top x + b_i \leq t_i, \; i = 1, \dots, n
\end{aligned}
$$

Idea of simplex algorithm



$Ax \leq b$

S - budget set

Idea of simplex:

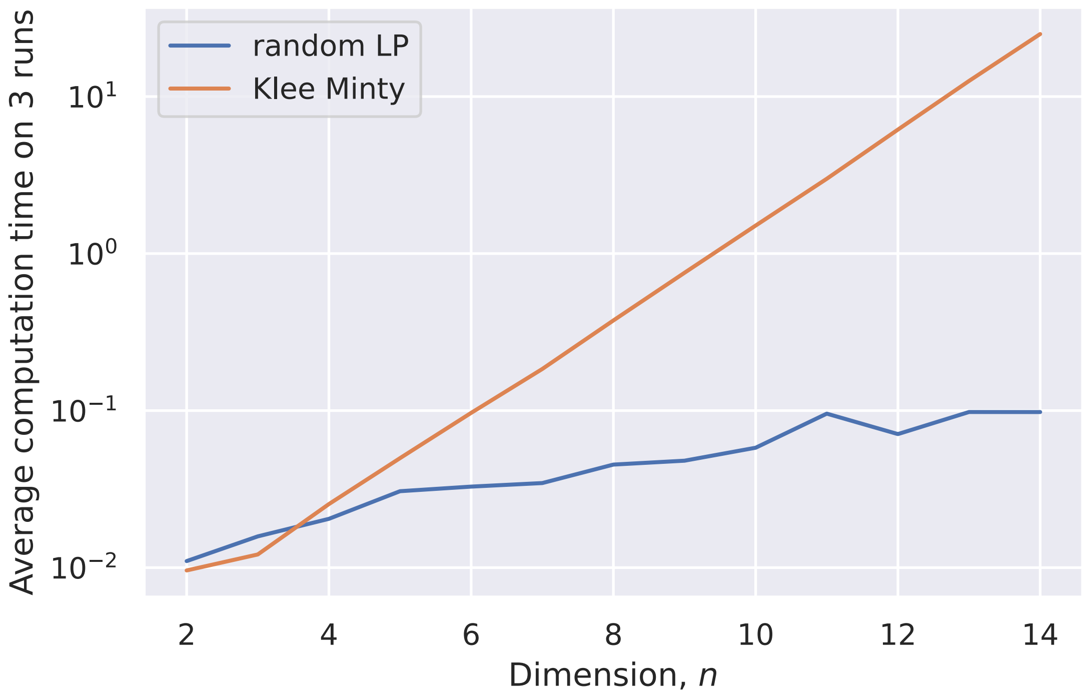
1. Start with some vertex
2. Select best vertex among available
3. Check the optimality

Convergence

Klee Minty example

In the following problem simplex algorithm needs to check $2^n - 1$ vertexes with $x_0 = 0$.

$$
\begin{aligned}
& \max_{x \in \mathbb{R}^n} 2^{n-1}x_1 + 2^{n-2}x_2 + \cdots + 2x_{n-1} + x_n \\
\text{s.t. } & x_1 \leq 5 \\
& 4x_1 + x_2 \leq 25 \\
& 8x_1 + 4x_2 + x_3 \leq 125 \\
& \ldots \\
& 2^n x_1 + 2^{n-1}x_2 + 2^{n-2}x_3 + \ldots + x_n \leq 5^n \qquad x \geq 0
\end{aligned}
$$



Summary

- A wide variety of applications could be formulated as the linear programming.
- Simplex algorithm is simple, but could work exponentially long.
- Khachiyan's ellipsoid method is the first to be proved running at polynomial complexity for LPs. However, it is usually slower than simplex in real problems.
- Interior point methods are the last word in this area. However, good implementations of simplex-based methods and interior point methods are similar for routine applications of linear programming.

Modern algorithms and soft on the old machines

Old algorithms and soft on the modern machines

Code

Open in Colab

Materials

- Linear Programming. in V. Lempitsky optimization course.
- Simplex method. in V. Lempitsky optimization course.
- Overview of different LP solvers
- TED talks watching optimization
- Overview of ellipsoid method
- Comprehensive overview of linear programming
- Converting LP to a standard form

1991 1998 (10x times) 2007 2012

CPLEX package

29 000 times improvement

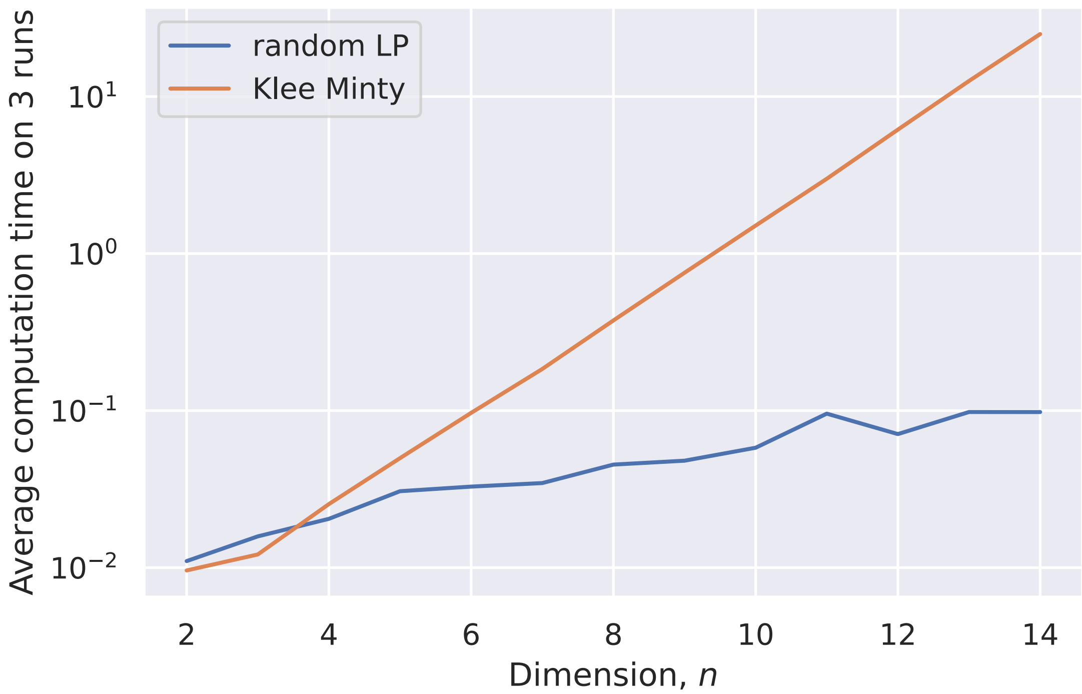
Gurobi 16 x times

Total acceleration:

469 800 x times faster

Hardware acceleration

~ 8000 x times.



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